Exercise Sheet #10

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P1. Let $(X, \|\bullet\|_X)$ and $(Y, \|\bullet\|_Y)$ be a normed spaces and $L : X \to Y$ a linear function. Prove that the following are equivalent:

(a) L is bounded, i.e. that there is $C > 0$ such that for each $x \in X$, $\|L(x)\|_Y \leq C\|x\|_X$,

(b) L is continuous,

(c) L is continuous at 0.

P2. Show that norm vector spaces are topological vector spaces.

P3. Let $I \subseteq \mathbb{R}$ be an interval, $\varphi : I \to \mathbb{R}$ a convex function. If $t \in \mathrm{Int}(I)$, then $\exists m \in \mathbb{R}$ s.t.

$$\varphi(s) \geq m(s - t) + \varphi(t), \quad \forall s \in I.$$

P4. Let $p, q \in (1, \infty)$ conjugate exponents and $f \in L^p$. Show that

$$\|f\|_p = \sup_{\|g\|_q \leq 1} \left| \int fg d\mu \right|.$$